Tutorial 4 10.08

Ruled surface (P85 ~ 89 in the text-book)

Def: We call a surface "ruled surface" if it has a parametrization

$$f(u,v) = c(u) + v \cdot X(u) \quad \text{s.t } \|X\| \equiv 1, \|X'\| \equiv 1$$

(*)

and $\langle c', X' \rangle = 0$, where $c(u)$, $X(u)$ are two C^2-curve in $\mathbb{R}^3$.

Rmk: (i) One may define a ruled surface as in P85 in the text-book "If it has a C^2-parametrization

$$f(u,v) = c(u) + v \cdot X(u)$$

where c, X are two curve, and X, X' never vanish". Then by change coordinate (reparametrization) and consider a new curve $\tilde{c} (= c - \langle c', x' \rangle x)$, we can obtain the form as (*).

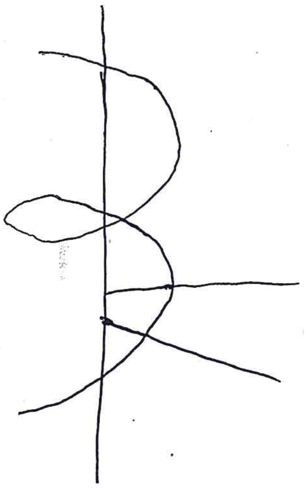
See Lemma 3.21 in the text-book.

(ii) Fix $u = u_0$, $f(u_0, v) = c(u_0) + v X(u_0)$ is a straight line passing through $c(u_0)$ in the direction $X(u_0)$ in the surface.

(iii) The ruled surface is covered by this family of straight lines.

eg. Helicoid

helix $\alpha(u) = (\alpha\cos u, \alpha\sin u, \beta u)$



Draw a line through $(0, 0, \beta u)$ and $(\alpha\cos u, \alpha\sin u, \beta u)$.

The surface swept out by these lines is a helicoid.

$$f(u,v) = (\alpha v\cos u, \alpha v\sin u, \beta u)$$

$$= (0, 0, \beta u) + v(\alpha\cos u, \alpha\sin u, 0)$$

Next we want to compute the 1st F.F., 2nd F.F., Gauss curvature mean curvature of a ruled surface

$$f(u,v) = c(u) + v\cdot X(u) \quad \text{with } \|X\| = \|X'\| \equiv 1,\ \langle c', X'\rangle =$$

1st F.F. $\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$ $\quad \begin{cases} f_u = c' + vX' \\ f_v = X \end{cases}$

$$E = \langle f_u, f_u\rangle = \langle c' + vX', c' + vX'\rangle = \|c'\|^2 + v^2.$$

$$F = \langle f_u, f_v\rangle = \langle c' + vX', X\rangle = \langle c', X\rangle$$

$$G = \langle f_v, f_v\rangle = \langle X, X\rangle = 1$$

Note that $X, X', X \times X'$ o.n.

$$c' = \langle c', X\rangle \cdot X + \langle c', X'\rangle \cdot X' + \langle c', X \times X'\rangle X \times X'$$

$$= F X + \lambda X \times X', \quad \lambda = \det(c', X, X').$$

$$\Rightarrow \quad \|c'\|^2 = F^2 + \lambda^2$$

matrix of 1st F.F.: $\begin{pmatrix} F^2 + \lambda^2 + v^2 & F \\ F & 1 \end{pmatrix}$

2nd F.F.

$$\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} l & m \\ m & n \end{pmatrix}$$

$$l = \langle f_{uu}, \nu \rangle, \quad m = \langle f_{uv}, \nu \rangle, \quad n = \langle f_{vv}, \nu \rangle$$

where ν = the Gauss map = the unit normal of f

$$= \frac{f_u \times f_v}{\|f_u \times f_v\|}$$

Note that $\|f_u \times f_v\|^2 = EG - F^2 = \det(g_{ij})$

by the so called Lagrange's identity:

$$\langle a \times b, c \times d\rangle = \langle a, c\rangle \cdot \langle b, d\rangle - \langle a, d\rangle \cdot \langle b, c\rangle.$$

$f_{uu} = c'' + v\cdot X''$, $f_{uv} = X'$, $f_{vv} = 0$

$$h_{11} = \frac{1}{\|f_u \times f_v\|} \langle f_u \times f_v , f_{uu} \rangle$$

$$= \frac{1}{\sqrt{\det g}} \langle (c' + vX') \times X , c'' + v\cdot X'' \rangle$$

$$\langle c' \times X + v X' \times X , F'X + FX' + \lambda' X \times X' + \lambda X \times X'' + v\cdot X'' \rangle$$

$$= \lambda F + \lambda' \underbrace{\langle c' \times X , X \times X' \rangle}_{①} + \lambda \underbrace{\langle c' \times X , X \times X'' \rangle}_{②} + v \underbrace{\langle c' \times X , X'' \rangle}_{③}$$

$$+ v\lambda' (-1) + v\lambda \underbrace{\langle X' \times X , X \times X'' \rangle}_{④} + (-1) v^2 J$$

$$\boxed{J = \det(X, X', X'')}$$

① $= 0 - 0 = 0$

② $= F\cdot\langle X, X''\rangle - \langle c', X''\rangle$

$= F\cdot\langle X, X''\rangle - \langle FX + \lambda X \times X', X''\rangle$

$= \cancel{0} - \lambda J$ $\qquad a \times b \times c = b\langle c, a\rangle - a\langle c, b\rangle$

③ $= \langle (FX + \lambda X \times X') \times X , X'' \rangle$

$= \langle \lambda X' - \lambda X \cdot 0 , X'' \rangle = 0$

④ $= 0 \cdot - 0 = 0.$

$$\Rightarrow \quad h_{11} = \frac{1}{(\lambda^2 + v^2)^{\frac{1}{2}}} \left(\lambda F - \lambda^2 J - v\lambda' - v^2 J \right)$$

$$h_{12} = \frac{1}{(\lambda^2+v^2)^{\frac{1}{2}}} \langle (c' + vX') \times X, \; X' \rangle$$

$$= \frac{\lambda}{(\lambda^2+v^2)^{\frac{1}{2}}}$$

$$h_{22} = 0.$$

$$K = \frac{\det(h_{ij})}{\det(g_{ij})} = - \frac{\lambda^2}{(\lambda^2+v^2)^2}$$

$$H = \frac{h_{11} g_{22} - 2 h_{12} g_{12} + h_{22} g_{11}}{2 \det(g_{ij})} = \frac{\lambda F - \lambda^2 J - v\lambda' - v^2 J - 2\lambda F}{2(\lambda^2+v^2)^{\frac{3}{2}}}$$

$$= - \frac{J v^2 + \lambda' v + \lambda(\lambda J + F)}{2(\lambda^2+v^2)^{\frac{3}{2}}}.$$

Exercise: Compute the 1st, 2nd F.F, K, H for a helicoid $f(u,v) = (0,0,\beta u) + v(\alpha \cos u, \alpha \sin u, 0)$

$$f(u,v) = \underbrace{(0,0,\beta u)}_{c(u)} + v\,\underbrace{(\cos u, \sin u, 0)}_{X(u)}$$

1st F.F.

$$E = \|c'\|^2 + v^2 = \beta^2 + v^2$$

$$F = \langle c', X\rangle = \langle (0,0,\beta), (\cos u, \sin u, 0)\rangle = 0$$

$$G = 1.$$

$$\det(g_{ij}) = \beta^2 + v^2.$$

2nd F.F.

$$h_{11} = \frac{1}{(\lambda^2+v^2)^{\frac{1}{2}}}\left(\lambda F - \lambda^2 J - v\lambda' - v^2 J\right)$$

$$\lambda = \det(c', X, X') = \det\begin{pmatrix} 0 & 0 & \beta \\ \cos u & \sin u & 0 \\ -\sin u & \cos u & 0 \end{pmatrix} = \beta \quad , \quad \lambda' = 0$$

$$J = \det(X, X', X'') = \det\begin{pmatrix} \cos u & \sin u & 0 \\ \cdot & \cdot & 0 \\ \cdot & \cdot & 0 \end{pmatrix} = 0$$

$$h_{11} = 0 \;;\; h_{12} = \frac{\lambda}{(\lambda^2+v^2)^{\frac{1}{2}}} = \frac{\beta}{(\beta^2+v^2)^{\frac{1}{2}}} \;,\; h_{22} = 0$$

$$K = -\frac{\beta^2}{(\beta^2+v^2)^2} \;,\; H = 0. \qquad (v \mapsto \alpha v)$$